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The b-chromatic number of mycielskian of cycles

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Abstract—A b-coloring of a graph G is a proper coloring of the vertices of G such that there exist a vertex in each color class joined to at least one vertex in each other color classes. The b-chromatic number of a graph G, denoted by j(G), is the maximal integer k such that G may have a b-coloring with k colors. The Mycielskian or Mycielski graph m(H) of a graph H with vertex set fv_1; v_2;:::; v_ng is a graph G obtained from H by adding n + 1 new vertices fu; u_1; u_2;:::; u_ng, joining u to each vertex u_i(1 i n) and joining u_i to each neighbour of v_i in H. In this paper we obtained the b-chromatic number of the mycielskian of cycles.

Index Terms—b-chromatic number, b-coloring, b-dominating set, mycielskian, cycle.

1 INTRODUCTION

The concept of b-chromatic number was introduced in 1999 by Irving and Manlove[6], who proved that determining $\varphi(G)$ is NP-hard in general and polynomial time solvable for trees. The b-chromatic number $\varphi(G)$ of a graph G is the largest positive integer k such that G admits a proper k-coloring in which every color class has a representative vertex which is adjacent to at least one vertex in each of the other color classes. Such a coloring is called a b-coloring and this representative vertex is known as the b-dominating vertex and the set of b-dominating vertex is known as the b-dominating set [3]. In [1] the b-coloring of cographs and P_4 sparse graphs is discussed. In [3] El Sahili and Kouider M obtained a general formula for the b-chromatic number of regular graphs. In [7] the b-coloring of Kneser graphs is discussed. In [10] Vernold Vivin J and Venkatachalam M obtained the b-chromatic number of corona of two graphs with same number of vertices. In this paper we obtained the b-chromatic number of mycielskian of cycles.

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2 B-CHROMATIC NUMBER OF A GRAPH

The b-chromatic number of a graph G is defined as follows.

Definition 2.1. The b-chromatic number $\varphi(G)$ of a graph G is the largest positive integer k such that G admits a proper

k-coloring in which every color class contains a vertex which is adjacent to at least one vertex in each of the other color classes.

Example 2.2.

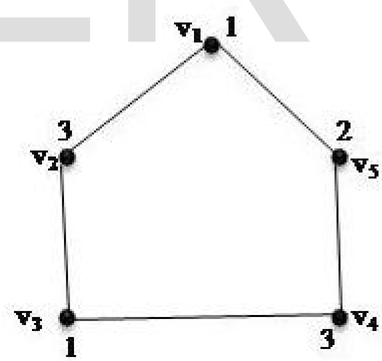

Figure 1: b-coloring of a graph with three colors

3 MYCIELSKIAN OF A GRAPH [5]

The mycielskian or mycielski graph of a graph G, denoted by $\mu(G)$ is defined as follows.

Definition 3.1. The Mycielskian or Mycielski graph $\mu(H)$ of

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a graph H with vertex set $\{v_1, v_2 \ldots. v_n\}$ is a graph G obtained from H by adding n + 1 new vertices $\{u_1, u_2 \ldots. u_n\}$ joining u to each vertex $u_i \; for \; 1 \le i \le n$ and joining u_i to each neighbour of v_i in H.

Example 3.2.

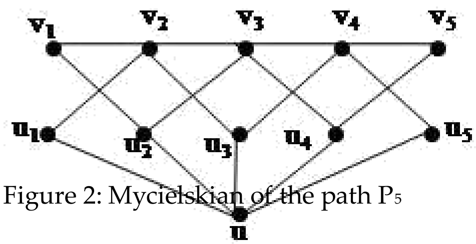

Figure 2: Mycielskian of the path P_5

4 B-CHROMATIC NUMBER OF MYCIEL-SKIAN OF CYCLE

The b-chromatic number of the mycielskian of cycle is given as follows

Theorem 4.1. The b-chromatic number of the my-cielskian of a cycle is

$$\varphi(\mu(C_n)) = \begin{cases} \varphi(C_n) + 1, & n \le 6 \\ \varphi(C_n) + 2, & n \ge 7 \end{cases}$$

Proof:

Let the vertex set of $\mu(C_n)$ be $\{v_1, v_2 \ldots. v_n\}$ and that of $\mu(C_n)$ be $\{v_1, v_2 \ldots. v_n\} \cup \{u_1, u_2 \ldots. u_n\} \cup \{u\}$ .Here $\{u_1, u_2 \ldots. u_n\}$ is set of n independent vertices in which each u_i is connected to every neighbours of v_i and the vertex u is connected to every u_i ; $1 \le i \le n$.

Case 1: $n \le 6$

Subcase 1: n = 3

Here $\varphi(C_n)$ = 3, so we Have to prove That $\varphi(\mu(C_n))$ = 4. On the Contrary Assume That $\varphi(\mu(C_n))$= 5, then there will be at least 5 vertices with degree at least 4. But here we have only 3 vertices with degree at least 4. So a b-coloring with 5 colors is not possible. A b-coloring with 4 colors can be obtained by assigning color i to v_i and u_i ; $1 \le i \le n$.and color 4 to u.

Subcase 2: n = 4

Here we have exactly 5 vertices, v_1, v_2, v_3, v_4, u with degree at least 4. Hence we can check the existence of a b-coloring with 5 colors. Assume that such a coloring is existing, then the b-dominating vertices will be v_1, v_2, v_3, v_4 and u. Now consider the vertices v_1 and v_3. The neighbours of v_1 are v_2, v_n, u_2, u_n, which are same as the neighbours of v_3. That is, here the two vertices v_1 and v_3 are having same neighbours. Hence a b-coloring with 5 colors is not possible. Because if we choose v_1 and v_3 as the b-dominating vertices then v_3 should be adjacent to a vertex which is having the color of v_1 and v_1 should be adjacent to a vertex which is having the color of v_3, but since these vertices have same neighbours we cannot assign the color of v_1 and color of v_3 to any of these neighbours. Next check the existence of a b-coloring with 4 colors. Here all the vertices are having degree at least 3. Now consider the set $\{v_1, v_2, v_3, v_4\}$. From this set we can choose only two b-dominating vertices because here v_1 and v_3 are having same neighbours and v_2 and v_4 are having same neighbours. Hence from this set we can select v_1,v_2 or v_2, v_3 or v_3, v_4 or v_4, v_1 as the b-dominating vertices. That is we can select any two adjacent v_i's as the b-dominating vertices. Next consider the set $\{u_1, u_2, u_3, u_4\}$. Here each vertices are having degree exactly 3. Here any two u_i's will have either same neighbours or it will have one common neighbour and two dis-tinct neighbours. The vertices with same neighbours cannot be choose as b-dominating vertices. So we can choose vertices with one common neighbour and two distinct neighbours as the b-dominating vertices. Thus we can select u_1, u_2 or u_2, u_3 or u_3, u_4 or u_4, u_1 as the b-dominating vertices. That is we can select$u_j$and $u_{j+1}, \; 1 \le i \le 4 \; , j + 1 = 1 \; if \; j + 1 > 4$ as the b-dominating vertices. Now suppose that v_i, v_{i+1} , u_j, u_{j+1} are the b-dominating vertices. Here u_j will be adjacent to either v_i or v_{i+1} but not to both. Similarly u_{j+1} will be adjacent to either v_i or v_{i+1} but not to both. Without loss of generality assume that u_j is adjacent to v_i and u_{j+1} is adjacent to v_{i+1}. Assign colors c_1, c_2, c_3, c_4 to $v_i, v_{i+1}, u_j, u_{j+1}$ respectively. Now u_j is having color c_3 and is adjacent to v_i with color c_1. To make the vertex u_j b-dominating, it should be adjacent to vertices with color c_2 and c_4. Now the neighbours of u_j are v_i, v_{i+2} and u. Here i + 2 = 1 if i + 2 > 4. Since v_{i+2} is adjacent to v_{i+1} we cannot assign color c_2 to v_{i+2}. So assign color c_4 to v_{i+2} and c_2 to u. Now consider u_{j+1}. The neighbours of u_{j+1} are v_{i+1}, v_{i+3} and u. Here i + 3 = 1 if i + 3 > 4. Here the vertex v_{i+1} and u have same color and the vertex u_{j+1} will be b-dominating if it is adjacent to vertices with color c_1 and c_4. But now the vertex u_{j+1} have only one uncolored neighbour. Hence u_{j+1} will not become a b-dominating vertex. Hence we cannot choose two vertices from the set $\{u_1, u_2, u_3, u_4\}$ as b-dominating vertices. So select one vertex u_j with color c_3 as a b-dominating vertex. Now this vertex will be adjacent to either v_i or v_{i+1}. Suppose that it is adjacent to v_i. As mentioned above, to make the vertex u_j b-dominating we have to assign color c_2 to u. But this will results in a b-coloring with 3 colors. Because here we cannot find a b-dominating vertex to assign color c_4. This means that we cannot choose any b-dominating vertex from the set $\{u_1, u_2, u_3, u_4\}$.

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Now select u as a b-dominating vertex. But here also we cannot construct a b-coloring with 4 colors. Because here we have only 3 b-dominating vertices,v_i, v_{i+1} and u. Using these three b-dominating vertices we cannot construct a b-coloring with 4 colors. Hence a b-coloring with 4 colors is not possible here. A b-coloring with 3 colors can be obtained by assigning color 1 to u, color 2 to v_1, v_3, u_1 and u_3 and color 3 to v_2, v_4, u_2 and u_4

Subcase 3: n = 5

$\varphi(C_n) = 3$. So we have to prove that

$\varphi(\mu(C_n)) = 4$ On the contrary assume that $\varphi(\mu(C_n)) = 5$. Then there should be at least 5 vertices with degree at least 4. Here the degree of each u_i: $1 \le i \le 5$ is 3. But the degree of each v_i; $1 \le i \le 5$ is 4 and the degree of u is 5. So here we can choose either {v_1, v_2, v_3, v_4, v_5} or any 4 vertices from the set {v_1, v_2, v_3, v_4, v_5} and the vertex u as the b-dominating vertices. Consider the first case. That is select {v_1, v_2, v_3, v_4, v_5} as the b-dominating set. The degree of each of these vertices is 4. So these 5 vertices will become b-dominating only if all the 4 neighbours each v_i receives distinct colors. Now assign color c_i to v_i; $1 \le i \le 5$. Now consider the vertex v_1. This vertex is having color c_1 and is adjacent to v_2, v_5, u_2 and u_5. Here v_2 and v_5 are having colors c_2 and c_5 respectively. To make the vertex v_1 b-dominating we have to assign colors c_3 and c_4 to u_2 and u_5. Here u_2 is adjacent to v_3 having color c_3. So we cannot assign color c_3 to u_2. So assign color c_3 to u_5 and c_4 to u_2. Thus v_1 becomes a b-dominating vertex. Next consider the neighbours of v_4. The neighbours of v_4 are v_3, v_5, u_3 and u_5. Here v_3 and u_5 are having color c_3. That is two neighbours of v_4 receives same color. Hence we cannot make the vertex v_4 b-dominating. Hence the vertices {v_1, v_2, v_3, v_4, v_5} will not become a set of b-dominating vertices. Now consider the second case. That is choose any four vertices from the set {v_1, v_2, v_3, v_4, v_5} and u as the b-dominating vertices. Assign color c_1 to u. Since each u_i; $1 \le i \le 5$ is adjacent to u, we cannot assign color c_1 to any of the u_i's. But the four vertices from the set {v_1, v_2, v_3, v_4, v_5} become b-dominating only if they adjacent to a vertex with color c_1.

But from the set five vertices, {v_1, v_2, v_3, v_4, v_5} we have selected only four vertices as the b-dominating vertices. So one vertex will remains here and we can assign color c_1 to this vertex. But still all the four b-dominating vertices will not be adjacent to this vertex. Only two of them will be adjacent to this vertex. Hence in this case also we cannot make a b-coloring with 5 colors. A b-coloring with 4 colors can be obtained by assigning color 1 to v_1 and v_4, color 2 to v_2 and u, color 3 to v_3 and v_5 and color 4 to u_1, u_2, u_3, u_4 and u_5.

Subcase 4: n = 6

Here also $\varphi(C_n) = 3$. So we have to prove that

$\varphi(\mu(C_n)) = 4$ On the contrary assume that $\varphi(\mu(C_n)) = 5$. Then there will be at least 5 vertices with degree at least 4. Here the vertices with degree at least 4 are v_1, v_2, v_3, v_4, v_5, v_6 and u. Note that the degree of v_1, v_2, v_3, v_4, v_5, v_6 is exactly 4 and that of u is 6. Now we can select any of the 5 vertices from the set {v_1, v_2, v_3, v_4, v_5, v_6, u} as the b-dominating vertices. Suppose that the vertex u is included in the b-dominating set. If we select u as a b-dominating vertex, then the remaining four b-dominating vertices will be from the set {v_1, v_2,…. v_6}. Let the color of u be c_1. Since all the u_i's are adjacent to u, we cannot assign color c_1 to these vertices. The four vertices from the set {v_1, v_2…. v_6} will be b-dominating if it is adjacent to a vertex with color c_1. Note that from this set of six vertices we have selected only four vertices as b-dominating and for the remaining two vertices we can assign color c_1. Hence choose the four b-dominating and the two non b-dominating vertices with color c_1 in such a way that all the b-dominating vertices are adjacent to one of the non b-dominating vertices with color c_1. For example if we choose the b-dominating vertices as v_1, v_3, v_4 and v_6 and the two non b-dominating vertices with color v_1 as v_2 and v_5, then here all the b-dominating vertices will be adjacent to either v_2 or v_5. That is all the b-dominating vertices are adjacent to a vertex with color c_1. Let the color of the b-dominating vertices v_1, v_3, v_4 and v_5 be c_2,c_3, c_4 and c_5 respectively. Now consider the vertex v_1. v_1 is having color c_2 and is adjacent to v_2, v_6, u_2 and u_6. Here v_2 and v_6 are having colors c_1 and c_5 respectively. To make the vertex v_1 b-dominating, it should be adjacent to vertices with colors c_3 and c_4. Here v_3 is having color c_3 and is adjacent to u_2. So we cannot assign color c_3 to u_2. So assign color c_4 to u_2 and c_3 to u_6.

Now consider the vertex v_3. v_3 is having color c_3 and is adjacent to v_2, v_4, u_2 and u_4. Here the color of u_2 and v_4 is c_4 and that of v_2 is c_1. To make the vertex v_3 b-dominating, it should be adjacent to vertices with colors c_2 and c_5. But now this vertex has only one uncolored neighbour. So we cannot make this vertex b-dominating. Thus a b-coloring with 5 colors is not possible here. Also we cannot make any b-coloring with 5 colors even if we choose the four b-dominating and the two non b-dominating vertices in any other order. Now suppose that the vertex u not included in the b-dominating set. Thus the five b-dominating vertices will be from the set {v_1, v_2, v_6}. As mentioned above here also we cannot make some vertices b-dominating. Hence a b-coloring with 5 colors is not possible here and a b-coloring with 4 colors can be obtained by assigning color 1 to v_1, u_3, u_4 and u_5, color 2 to v_2, v_5 and u, color 3 to v_3, v_6 and u_6 and color 4 to v_4, u_1 and u_2.

Case 2: $n \ge 7$

$\varphi(C_n) = 3$. So we have to prove that $\varphi(\mu(C_n)) = 5$ On the contrary assume that $\varphi(\mu(C_n)) = 6$. Then there will be at least 6 vertices with degree at least 5. But here we have only one vertex with degree at least 5. So a b-coloring with 6 colors is not possible here. A b-coloring with 5 colors can be obtained by assigning color 1 to v_2, u_4, u_5 and u_{n-1}, color 2 to v_3, v_6 and v_{n-1}, color 3 to v_4, v_1 and u, color 4 to v_5, u_1 and u_2 and color 5 to v_n, u_n, u_3 and u_4. For the remaining u_i's and v_i's assign any of the color from the list {1, 2, 3, 4. 5}\ {3} in a proper way. Now this is a b-coloring with 5 colors and here the b-dominating vertices are v_2, v_3, v_4, v_5 and v_n.

5. CONCLUSION

In this paper we obtained the b-chromatic number of the mycielskian of cycles.

REFERENCES

[1] Bonomo, F., Duran, G., Maffray, F., Marenco, J., Valencia-Pabon, M.: On the b-coloring of cographs and P_4-spars graphs. Graphs Combin. 25, 153-157(2009)

[2] Djamel, G., Veronique, E., Frank, L., Hubert, E.: Graph b-coloring for automatic recognition of documents.IEEE/ICDAR,72(2009)

[3] El Sahili, A., Kouider, M.: About b-coloring of regular graphs. Res. Rep. 1432, LRI, Univ. Orsay, France(2006)

[4] Elghazel, H., Yoshida, T., Deslandres, V., Hacid, M.S., Dus-sauchoy, A.: A new greedy algorithm for improving b-coloirng clustering. Proceedings of GbR2007,Springer Verlag, 228-239(2007)